ISSN 1066-369X, Russian Mathematics, 2018, Vol. 62, No. 1, pp. 58–64. © Allerton Press, Inc., 2018.
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Computable Linear Orders and the Ershov Hierarchy

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Received October 19, 2016

Abstract—We give the collection of relations on computable linear orders. For any natural number n, the degree spectrum of such relations of some computable linear orders contains exactly all n-computable enumerable degrees. We also study interconnections of these relations among themselves.

DOI: 10.3103/S1066369X18010085

Keywords: *computable linear orders, degree spectrum of relations, n-computable enumerable degrees.*

INTRODUCTION

This paper is devoted to the study of computable linear orders enriched with predicates. For background on computability theory, see R. Soare [1], and on linear orders, see J. Rosenstein [2].

A linear order $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$ is called *computable* (Π_1-), if the univers of L and the order relation $<_{\mathcal{L}}$ are both computable (Π_1-, respectively). The successor and the block relations are natural relations in the theory of computable linear orders. Recall that the successor relation on linear order $\mathcal{L}$ is $S^0_{\mathcal{L}}(x, y) \leftrightharpoons |(x, y)_{\mathcal{L}}| = 0$, the block relation is $F_{\mathcal{L}}(x, y) \leftrightharpoons \exists n \in \omega\, (|(x, y)_{\mathcal{L}}| = n)$, where the set $(x, y)_{\mathcal{L}} = \{z \mid x <_{\mathcal{L}} z <_{\mathcal{L}} y\}$ is called an *interval*.

These relations on linear orders were objects of the study of different authors. M. Moses [3, 4] showed that a linear order has an 1-decidable presentation (a structure is 1-decidable, if all its 1-quantifier formulas are uniformly computable) if and only if it has a computable presentation with computable successors. Also in [3, 4] the block relation was studied and it was showed that the block relation of a computably categorical 1-decidable linear order is computable. J. Remmel [5] showed that a computable linear order is computably categorical if and only if it has only finitely many successors.

A. N. Frolov [6] and, independently, A. Montalban [7] proved that a linear order has a low presentation if and only if it has a $\mathbf{0}'$-computable presentation with $\mathbf{0}'$-computable successors. Thus, the successor relation is often used to study low linear orders, and, as shown by A. N. Frolov [8], and low$_2$ linear orders.

A. N. Frolov [9] and P. Alaev, J. Thurber, A. Frolov [10] used the notion of the successor relation in the study of algorithmic properties of quasidiscrete linear orders.

A. N. Frolov [11] showed that the degree spectrum of the successor relation of a computable non-η-like linear order is closed upwards in computably enumerable (c. e.) degrees (a linear order is called *η-like*, if it does not contain infinite blocks). Recall that the degree spectrum of a relation $P_{\mathcal{L}}$ of a computable linear order $\mathcal{L}$ is called the class $\mathrm{DgSp}_{\mathcal{L}}(P) = \{\deg_T(P_{\mathcal{R}}) \mid (\exists \mathcal{R} \cong \mathcal{L})\, \mathcal{R} \equiv_T \varnothing\}$. A. N. Frolov, V. Harizanov, and J. Chubb [12] showed that the degree spectrum of the successor relation on a computable linear order of a special type is closed upwards in c. e. degrees. Also A. N. Frolov [13] find some examples of the spectra of the successor relation.

R. I. Bikmukhametov [14–17] studied algorithmic independence of the relations above and some other natural relations on computable linear orders.

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